

Influence of the bound states in the Neumann Laplacian in a thin waveguide

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Abstract

We study the Neumann Laplacian operator $-\Delta_\Omega^N$ restricted to a twisted waveguide Ω . The goal is to find the effective operator when the diameter of Ω tends to zero. However, when Ω is “squeezed” there are divergent eigenvalues due to the transverse oscillations. We show that each one of these eigenvalues influences the action of the effective operator in a different way. In the case where Ω is periodic and sufficiently thin, we find information about the absolutely continuous spectrum of $-\Delta_\Omega^N$ and the existence and location of band gaps in its structure.

1 Introduction and main results

The Laplacian operator in a thin set with Neumann boundary conditions has been studied in various situations [7, 8, 11, 12]. In particular, let $-\Delta_\Omega^N$ be the Neumann Laplacian operator restricted to a thin waveguide Ω in \mathbb{R}^3 . An interesting question is to study the behavior of $-\Delta_\Omega^N$ when the diameter of Ω tends to zero and to find the effective operator T in this process. Since Ω shrinks to a spatial curve, it is natural to associate T with a one-dimensional operator. In fact, it is known that T is the one-dimensional Neumann Laplacian operator; in this case, its action is given by $w \mapsto -w''$. See, for example, [12]. This result holds even if Ω is a twisted or a bent waveguide, i.e., the geometry of Ω does not influence the action of the effective operator.

In this work we study $-\Delta_\Omega^N$ in the case where Ω is a twisted waveguide. As a first goal we study its behavior as the diameter of Ω tends to zero. In this process there are divergent eigenvalues due to the transverse oscillations in Ω . Our strategy shows that each one of these eigenvalues influences the action of the effective operator in a different way. Namely, the twisted effect influences directly its action, see (9), (10), (11) and (12) in this Introduction. The second goal of this work is to consider the case where Ω is periodic in the sense that the twisted effect varies periodically. In the case that Ω is small enough, we find information about the absolutely continuous spectrum of the Neumann Laplacian and the existence and location of band gaps in its structure. In the next paragraphs, we explain the model and provide details of our main results.

Let $I = \mathbb{R}$ or $I = (a, b)$ a bounded interval in \mathbb{R} . Pick $S \neq \emptyset$ an open, bounded, smooth and connected subset of \mathbb{R}^2 ; denote by $y := (y_1, y_2)$ an element of S . Let $\alpha : \bar{I} \rightarrow \mathbb{R}$ be a C^2 function; we suppose that $\alpha', \alpha'' \in L^\infty(I)$ and $\alpha(0) = 0$ if $I = \mathbb{R}$, or $\alpha(a) = 0$ if $I = (a, b)$. For each $\varepsilon > 0$ small enough, we define the thin twisted waveguide

$$\Omega_\varepsilon^\alpha := \{\Gamma_\varepsilon^\alpha(s)\mathbf{x}^t, \mathbf{x} = (s, y) \in I \times S\},$$

where

$$\Gamma_\varepsilon^\alpha(s) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon \cos \alpha(s) & -\varepsilon \sin \alpha(s) \\ 0 & \varepsilon \sin \alpha(s) & \varepsilon \cos \alpha(s) \end{pmatrix}. \quad (1)$$

Let $-\Delta_{\Omega_\varepsilon}^N$ be the Neumann Laplacian operator on $L^2(\Omega_\varepsilon)$, i.e., the self-adjoint operator associated with the quadratic form

$$\tilde{b}_\varepsilon(\psi) = \int_{\Omega_\varepsilon} |\nabla \psi|^2 d\vec{x}, \quad \text{dom } \tilde{b}_\varepsilon = H^1(\Omega_\varepsilon). \quad (2)$$

Since we are going to use the Γ -convergence technique (see Appendix A.2 and [4]), our analysis is based on the study of the sequence $(\tilde{b}_\varepsilon)_\varepsilon$. To simplify the calculations, it is convenient a change of variables. Using the change of coordinates described in Section 2, the quadratic form \tilde{b}_ε becomes

$$\hat{b}_\varepsilon(\psi) = \int_Q \left(|\psi' + \langle \nabla_y \psi, Ry \rangle \alpha'(s)|^2 + \frac{|\nabla_y \psi|^2}{\varepsilon^2} \right) ds dy, \quad (3)$$

$\text{dom } \hat{b}_\varepsilon = H^1(Q)$, $Q := I \times S$. Here, $\psi' := \partial \psi / \partial s$, $\nabla_y \psi := (\partial \psi / \partial y_1, \partial \psi / \partial y_2)$ and R is the rotation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Denote by $-\hat{\Delta}^\varepsilon$ the self-adjoint operator associated with \hat{b}_ε .

When the waveguide is “squeezed”, i.e., $\varepsilon \rightarrow 0$, $-\Delta_{\Omega_\varepsilon}^N$ presents divergent eigenvalues due to the transverse oscillations in $\Omega_\varepsilon^\alpha$; one can see this by the presence of the term $(1/\varepsilon^2) \int_Q |\nabla_y \psi|^2 ds dy$ in (3). To control this divergent energies, we will take the following strategy: let $-\Delta_S^N$ be the Neumann Laplacian operator restricted to S , i.e., the self-adjoint operator associated with the quadratic form $u \mapsto \int_S |\nabla_y u|^2 dy$, $u \in H^1(S)$. Denote by λ_n the n th eigenvalue of $-\Delta_S^N$ and by u_n the corresponding normalized eigenfunction, i.e.,

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \dots, \quad -\Delta_S^N u_n = \lambda_n u_n, \quad n = 1, 2, 3, \dots.$$

We assume that each eigenvalue λ_n is simple; note that u_1 is a constant function.

Fixed $n \in \mathbb{N}$, our strategy is to study the sequence

$$\hat{b}_n^\varepsilon(\psi) := \hat{b}_\varepsilon(\psi) - \frac{\lambda_n}{\varepsilon^2} \|\psi\|_{L^2(Q)}^2,$$

$\text{dom } \hat{b}_n^\varepsilon := H^1(Q)$. Denote by \hat{T}_n^ε the self-adjoint operator associated with \hat{b}_n^ε ; this can be done since each quadratic form \hat{b}_n^ε is closed and lower bounded in $L^2(Q)$. Namely, $\hat{T}_n^\varepsilon = -\hat{\Delta}^\varepsilon - (\lambda_n/\varepsilon^2)\mathbf{1}$; $\mathbf{1}$ denotes the identity operator.

It is usual in the literature to consider only the case $n = 1$, i.e., since $\lambda_1 = 0$, to study directly the sequence of quadratic forms $\hat{b}_\varepsilon(\psi)$. The idea to consider $n \neq 1$ is based on [5]; the author considered the Dirichlet Laplacian operator restricted to a thin waveguide with the goal of finding the effective operator. In that case, the action of the effective operator is the same for $n = 1$ or $n \neq 1$ and depends on the geometry of the waveguide.

Now, for each $n \in \mathbb{N}$, consider the closed subspaces

$$\mathcal{L}_n := \{w(s)u_n(y) : w \in L^2(I)\} \quad \text{and} \quad \mathcal{K}_n := \{w(s)u_n(y) : w \in H^1(I)\}$$

of $L^2(Q)$ and $H^1(Q)$, respectively. We have the decompositions

$$L^2(Q) = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \dots, \quad H^1(Q) = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3 \oplus \dots,$$

and each \mathcal{K}_n is a dense subspace of \mathcal{L}_n .

Let $T_1 w := -w''$ be the one-dimensional Laplacian operator with domain $\text{dom } T_1 = H^2(\mathbb{R})$ if $I = \mathbb{R}$, or $\text{dom } T_1 = \{w \in H^2(I) : w'(a) = w'(b) = 0\}$ if $I = (a, b)$. Denote by $\mathbf{0}$ the null operator on the subspace \mathcal{L}_1^\perp . In the particular case $n = 1$, it is known that $\hat{T}_1^\varepsilon \approx T_1 \oplus \mathbf{0}$, as $\varepsilon \rightarrow 0$; see [12]. As already commented, we can note that the effective operator in this situation does not depend on the geometry of the waveguide.

The main goal of this work is to study the sequence $(\hat{T}_n^\varepsilon)_\varepsilon$ (for each $n > 1$ fixed), and to characterize the effective operator in the limit $\varepsilon \rightarrow 0$. However, some adjustments will be necessary so that the limit exists in some sense. The interesting point in this situation is that we find an effective operator that depends on the geometry of the waveguide. To our knowledge, this fact is not known yet.

In order to study the sequence $(\hat{T}_n^\varepsilon)_\varepsilon$, it will be necessary some considerations. If $v(s, y) = w(s)u_j(y)$ with $w \in H^1(I)$, some calculations show that

$$\begin{aligned} \hat{b}_n^\varepsilon(v) &= \int_Q |w' u_j + \langle \nabla_y u_j, Ry \rangle \alpha'(s) w|^2 ds dy + \frac{1}{\varepsilon^2} \int_Q (|\nabla_y u_j|^2 - \lambda_n |u_j|^2) |w|^2 ds dy \\ &= \int_Q |w' u_j + \langle \nabla_y u_j, Ry \rangle \alpha'(s) w|^2 ds dy + \frac{(\lambda_j - \lambda_n)}{\varepsilon^2} \|w\|_{L^2(I)}^2, \end{aligned}$$

i.e., for $j < n$,

$$\lim_{\varepsilon \rightarrow 0} \hat{b}_n^\varepsilon(v) = -\infty. \quad (4)$$

Thus, the sequence $(\hat{b}_n^\varepsilon(v))_\varepsilon$ is not bounded from below. Therefore, to study $(\hat{b}_n^\varepsilon)_\varepsilon$, it will be necessary to exclude some vectors of the domains $\text{dom } \hat{b}_n^\varepsilon$. Based on (4), the procedure for this problem is as follows. We define the Hilbert spaces

$$\mathcal{H}_n := \begin{cases} L^2(Q), & n = 1, \\ (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{n-1})^\perp, & n = 2, 3, \dots, \end{cases} \quad (5)$$

equipped with the norm of $L^2(Q)$. Then, we consider the sequence of quadratic forms acting in \mathcal{H}_n ;

$$\bar{b}_n^\varepsilon(\psi) = \int_Q \left(|\psi' + \langle \nabla_y \psi, Ry \rangle \alpha'(s)|^2 + \frac{1}{\varepsilon^2} |\nabla_y \psi|^2 \right) ds dy, \quad (6)$$

$\text{dom } \bar{b}_n^\varepsilon = H^1(Q) \cap \mathcal{H}_n$, and we denote by $-\Delta_n^\varepsilon$ the self-adjoint operator on \mathcal{H}_n associated with it. Finally, define

$$b_n^\varepsilon(\psi) := \bar{b}_n^\varepsilon(\psi) - \frac{\lambda_n}{\varepsilon^2} \|\psi\|_{\mathcal{H}_n}^2, \quad (7)$$

$\text{dom } b_n^\varepsilon := H^1(Q) \cap \mathcal{H}_n$. Denote by T_n^ε the self-adjoint operator associated with b_n^ε which is a positive and closed quadratic form; T_n^ε acts in the Hilbert space \mathcal{H}_n . Namely, $T_n^\varepsilon = -\Delta_n^\varepsilon - (\lambda_n/\varepsilon^2)\mathbf{1}$. Then, we are going to study the sequence $(T_n^\varepsilon)_\varepsilon$ instead of $(\hat{T}_n^\varepsilon)_\varepsilon$.

Let b_n be the one-dimensional quadratic form

$$b_n(w) := b_n^\varepsilon(wu_n) = \int_Q |w' u_n + \langle \nabla_y u_n, Ry \rangle \alpha'(s) w|^2 ds dy, \quad (8)$$

$\text{dom } b_n = H^1(I)$. In fact, b_n is obtained by the restriction of b_n^ε to the space \mathcal{K}_n . Denote by T_n the self-adjoint operator associated with b_n .

For each $n \in \mathbb{N}$, define the constants

$$C_n^1(S) := \int_S |\langle \nabla_y u_n, Ry \rangle|^2 dy, \quad C_n^2(S) := \int_S u_n \langle \nabla_y u_n, Ry \rangle dy, \quad (9)$$

and the real potential

$$V_n(s) := C_n^1(S)(\alpha'(s))^2 - C_n^2(S)\alpha''(s). \quad (10)$$

By considerations of Appendix A.1,

$$T_n w = -w'' + V_n(s)w, \quad (11)$$

where $\text{dom } T_n = H^2(\mathbb{R})$ if $I = \mathbb{R}$ and,

$$\text{dom } T_n = \left\{ w \in H^2(I) : \begin{array}{l} w'(a) = -C_n^2(S)\alpha'(a)w(a) \\ w'(b) = -C_n^2(S)\alpha'(b)w(b) \end{array} \right\} \quad (12)$$

if $I = (a, b)$. In the latter, we have the Robin conditions in $\text{dom } T_n$.

Now, we present the first result of this work.

Theorem 1. (A) For each $n \in \mathbb{N}$ fixed, the sequence of self-adjoint operators T_n^ε converges in the strong resolvent sense to T_n in \mathcal{L}_n , as $\varepsilon \rightarrow 0$. That is,

$$\lim_{\varepsilon \rightarrow 0} R_{-\lambda}(T_n^\varepsilon)\zeta = R_{-\lambda}(T_n)P\zeta, \quad \forall \zeta \in \mathcal{H}_n, \forall \lambda > 0,$$

where P is the orthogonal projection onto \mathcal{L}_n .

(B) In addition, suppose $I = (a, b)$ a bounded interval. Denote by $\mu_j(\varepsilon)$ (resp. μ_j) the j th eigenvalue of $-\Delta_n^\varepsilon$ (resp. T_n) counted according to its multiplicity. Then, for each $j \in \mathbb{N}$,

$$\mu_j = \lim_{\varepsilon \rightarrow 0} \left(\mu_j(\varepsilon) - \frac{\lambda_n}{\varepsilon^2} \right). \quad (13)$$

In the next paragraphs we treat the periodic case.

Consider the twisted waveguide $\Omega_\varepsilon^\alpha$ in the particular case where $I = \mathbb{R}$ and $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 and periodic function, i.e., there exists $L > 0$ so that $\alpha(s+L) = \alpha(s)$, for all $s \in \mathbb{R}$. The second goal of this work is to find spectral properties of $-\Delta_n^\varepsilon$ in this situation. We have

Theorem 2. For each $n \in \mathbb{N}$ and for each $E > 0$, there exists $\varepsilon_E > 0$ so that the spectrum of $-\Delta_n^\varepsilon$ is absolutely continuous in the interval $[0, E + \lambda_n/\varepsilon^2]$, for all $\varepsilon \in (0, \varepsilon_E)$.

Theorem 3. Suppose that $V_n(s)$ is not a constant function in $[0, L]$. For each $n \in \mathbb{N} \setminus \{1\}$, there exist $j \in \mathbb{N}$ and $\varepsilon_j > 0$, so that, for all $\varepsilon \in (0, \varepsilon_j)$, the spectrum of the operator $-\Delta_n^\varepsilon$ has at least one gap.

Furthermore, in Theorem 6 in Section 4.4, we find a location in $\sigma(-\Delta_n^\varepsilon)$ where Theorem 3 holds true.

The proof of Theorem 3 is based on the fact that $V_n(s)$ is not a constant function in $[0, L]$. Due to this reason, we eliminate the case $n = 1$ since $V_1(s) \equiv 0$.

This work is separated as follows. In Section 2 we perform the change of variables to obtain (3) and in Section 3 we prove Theorem 1. Section 4 is dedicated to the periodic case and is separated in subsections. In Subsection 4.1 we present some preliminary results and in Subsection 4.2 we prove Theorem 2. Subsection 4.3 is dedicated to prove Theorem 3 and in Subsection 4.4 we study the location of band gaps. Along the text, the symbol K is used to denote different constants and it never depends on θ .

2 Geometry of the domain

Recall the quadratic form \tilde{b}_ε defined by (2). In this section we perform a usual change of variables so that the domain $\text{dom } \tilde{b}_\varepsilon$ becomes independent of ε . Then, consider the mapping

$$F_\varepsilon : \begin{array}{ccc} Q & \rightarrow & \Omega_\varepsilon^\alpha \\ (s, y_1, y_2) & \mapsto & \Gamma_\varepsilon^\alpha(s)(s, y_1, y_2)^t \end{array},$$

where $\Gamma_\varepsilon^\alpha(s)$ is given by (1); F_ε will be a (global) diffeomorphism for $\varepsilon > 0$ small enough.

In the new variables the domain $\text{dom } \tilde{b}_\varepsilon$ turns to be $H^1(Q)$. On the other hand, the price to be paid is a nontrivial Riemannian metric $G = G_\varepsilon^\alpha$ which is induced by F_ε i.e., $G = (G_{ij})$, $G_{ij} = \langle e_i, e_j \rangle$, $1 \leq i, j \leq 3$, where $e_1 = \partial F_\varepsilon / \partial s$, $e_2 = \partial F_\varepsilon / \partial y_1$, and $e_3 = \partial F_\varepsilon / \partial y_2$. Some calculations show that

$$J = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & -\varepsilon \alpha'(s) \langle z_\alpha^\perp(s), y \rangle & \varepsilon \alpha'(s) \langle z_\alpha(s), y \rangle \\ 0 & \varepsilon \cos \alpha(s) & \varepsilon \sin \alpha(s) \\ 0 & -\varepsilon \sin \alpha(s) & \varepsilon \cos \alpha(s) \end{pmatrix},$$

where

$$z_\alpha(s) := (\cos \alpha(s), -\sin \alpha(s)), \quad z_\alpha^\perp(s) := (\sin \alpha(s), \cos \alpha(s)).$$

The inverse matrix of J is given by

$$J^{-1} = \begin{pmatrix} 1 & \alpha'(s)y_2 & -\alpha'(s)y_1 \\ 0 & (\cos \alpha(s))/\varepsilon & -(\sin \alpha(s))/\varepsilon \\ 0 & (\sin \alpha(s))/\varepsilon & (\cos \alpha(s))/\varepsilon \end{pmatrix}.$$

Note that $JJ^t = G$ and $\det J = |\det G|^{1/2} = \varepsilon^2 > 0$. Thus, F_ε is a local diffeomorphism. In the case that F_ε is injective (for this, just consider $\varepsilon > 0$ small enough), a global diffeomorphism is obtained.

Introducing the unitary transformation

$$U_\varepsilon : \begin{array}{ccc} L^2(\Omega_\varepsilon^\alpha) & \rightarrow & L^2(Q) \\ \phi & \mapsto & \varepsilon \phi \circ F_\varepsilon \end{array},$$

we obtain the quadratic form

$$\begin{aligned} \hat{b}_\varepsilon(\psi) &:= \tilde{b}_\varepsilon(U_\varepsilon^{-1}\psi) = \|J^{-1}\nabla\psi\|_{L^2(Q)}^2 \\ &= \int_Q \left(|\psi' + \langle \nabla_y \psi, Ry \rangle \alpha'(s)|^2 + \frac{|\nabla_y \psi|^2}{\varepsilon^2} \right) ds dy, \end{aligned}$$

$\text{dom } \hat{b}_\varepsilon = H^1(Q)$. Recall R is the rotation matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $-\hat{\Delta}_\varepsilon$ denotes the self-adjoint operator associated with \hat{b}_ε . We have $-\hat{\Delta}_\varepsilon \psi = U_\varepsilon(-\Delta_{\Omega_\varepsilon^\alpha}^N)U_\varepsilon^{-1}\psi$, where $\text{dom } (-\hat{\Delta}_\varepsilon) = U_\varepsilon(\text{dom } (-\Delta_{\Omega_\varepsilon^\alpha}^N))$.

3 Preliminary results and poof of Theorem 1

This section is dedicated to the proof of Theorem 1. The strategy is based on the study of the sequence $(b_n^\varepsilon)_\varepsilon$ (see (7) in the Introduction) and some preliminary results will be necessary. We start with some considerations. Denote by $[u_1, u_2, \dots, u_k]$ the subspace of $L^2(S)$ generated by $\{u_1, u_2, \dots, u_k\}$. Since the subspace $\mathcal{W}_k := [u_1, u_2, \dots, u_k]^\perp$ is

invariant under the operator $-\Delta_S^N$, the restriction $-\Delta_S^N|_{\mathcal{W}_k}$ is well defined and its first eigenvalue is λ_{k+1} . Denote by q_k the quadratic form associated with $-\Delta_S^N|_{\mathcal{W}_k}$. We have

$$q_k(v) \geq \lambda_{k+1} \|v\|_{L^2(S)}^2, \quad \forall v \in \mathcal{W}_k \cap H^1(S). \quad (14)$$

To study the sequence $(b_n^\varepsilon)_\varepsilon$ we are going to use the Γ -convergence technique; see Appendix A.2. Then, it is necessary to extend b_n^ε on \mathcal{H}_n by setting (we denote by the same symbol)

$$b_n^\varepsilon(v) = \begin{cases} b_n^\varepsilon(v), & \text{if } v \in \text{dom } b_n^\varepsilon, \\ +\infty, & \text{otherwise.} \end{cases}$$

In a similar way, we extend b_n on \mathcal{H}_n ;

$$b_n(v) = \begin{cases} b_n(w), & \text{if } v = wu_n \text{ with } w \in \text{dom } b_n, \\ +\infty, & \text{otherwise;} \end{cases}$$

recall the definition of b_n by (8) in the Introduction.

Lemma 1. *If $v_\varepsilon \rightharpoonup v$ in \mathcal{H}_n and $(b_n^\varepsilon(v_\varepsilon))_\varepsilon$ is a bounded sequence, then $(v'_\varepsilon)_\varepsilon$ and $(\nabla_y v_\varepsilon)_\varepsilon$ are bounded sequences in \mathcal{H}_n . Furthermore, $v \in H^1(Q)$ and there exists a subsequence of $(v_\varepsilon)_\varepsilon$, denoted by the same symbol $(v_\varepsilon)_\varepsilon$, so that $v'_\varepsilon \rightharpoonup v'$ and $\nabla_y v_\varepsilon \rightharpoonup \nabla_y v$.*

Proof. Since $(v_\varepsilon)_\varepsilon$ and $(b_n^\varepsilon(v_\varepsilon))_\varepsilon$ are bounded sequences, there exists a number $K > 0$ so that

$$\limsup_{\varepsilon \rightarrow 0} \int_Q |v'_\varepsilon + \langle \nabla_y v_\varepsilon, Ry \rangle \alpha'(s)|^2 ds dy \leq \limsup_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) < K,$$

and

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \int_Q |\nabla_y v_\varepsilon|^2 ds dy \\ &= \limsup_{\varepsilon \rightarrow 0} \left(\int_Q (|\nabla_y v_\varepsilon|^2 - \lambda_n |v_\varepsilon|^2) ds dy + \int_Q \lambda_n |v_\varepsilon|^2 ds dy \right) \\ &\leq \limsup_{\varepsilon \rightarrow 0} K \varepsilon^2 + \limsup_{\varepsilon \rightarrow 0} \int_Q \lambda_n |v_\varepsilon|^2 ds dy < K. \end{aligned} \quad (15)$$

These estimates, and the fact that α' and Ry are bounded functions, show that $(v'_\varepsilon)_\varepsilon$ and $(\nabla_y v_\varepsilon)_\varepsilon$ are bounded sequences in $L^2(Q)$. Therefore, $(v_\varepsilon)_\varepsilon$ is a bounded sequence in $H^1(Q)$. Thus, there exists $\psi \in H^1(Q)$ and a subsequence of $(v_\varepsilon)_\varepsilon$, also denoted by $(v_\varepsilon)_\varepsilon$, so that $v_\varepsilon \rightharpoonup \psi$ in $H^1(Q)$ (recall that this Hilbert space is reflexive). As $v_\varepsilon \rightharpoonup v$ in \mathcal{H}_n , it follows that $v = \psi$, $v'_\varepsilon \rightharpoonup v'$, $\nabla_y v_\varepsilon \rightharpoonup \nabla_y v$ in \mathcal{H}_n and $v \in H^1(Q)$. \square

Lemma 2. *If $v_\varepsilon \rightarrow v$ in \mathcal{H}_n and there exists the limit $\lim_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) < +\infty$, then $v(s, y) = w(s)u_n(y)$ with $w \in H^1(I)$ (i.e., $v \in \mathcal{K}_n$).*

Proof. By Lemma 1, passing to a subsequence if necessary, $\nabla_y v_\varepsilon \rightharpoonup \nabla_y v$ in $L^2(Q)$. By weak lower semi-continuity of the L^2 -norm, inequality (15) and from the strong convergence of $(v_\varepsilon)_\varepsilon$, we have

$$\int_Q |\nabla_y v|^2 ds dy \leq \liminf_{\varepsilon \rightarrow 0} \int_Q |\nabla_y v_\varepsilon|^2 ds dy \leq \limsup_{\varepsilon \rightarrow 0} \lambda_n \int_Q |v_\varepsilon|^2 ds dy = \lambda_n \int_Q |v|^2 ds dy.$$

Now, define the function $f_n(s) := \int_S (|\nabla_y v(s, y)|^2 - \lambda_n |v(s, y)|^2) dy$. The latter inequalities show that $f_n(s) \leq 0$. However, (14) ensures that $f_n(s) \geq 0$. Then, $f_n = 0$ a.e.. We conclude that $v(s, \cdot) \in \mathcal{W}_{n-1} \cap H^1(S)$, and $v(s, \cdot)$ is an eigenfunction of the operator $-\Delta_S^N|_{\mathcal{W}_{n-1}}$ whose eigenvalue associated is λ_n . As λ_n is simple, $v(s, \cdot)$ is proportional to u_n . Thus, we can write $v(s, y) = w(s)u_n(y)$ with $w \in H^1(I)$, since $v \in H^1(Q)$. \square

Proposition 1. *For each $n \in \mathbb{N}$, the sequence of quadratic forms b_n^ε strongly Γ -converges to b_n , as $\varepsilon \rightarrow 0$.*

Proof. We have to prove the items (i) and (ii) according to the definition of strong Γ -convergence in Appendix A.2.

Let $v \in \mathcal{H}_n$ and $v_\varepsilon \rightarrow v$ in \mathcal{H}_n . If $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = +\infty$, then $b_n(v) \leq \liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon)$. Now, assume that $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) < +\infty$. Passing to a subsequence if necessary, we can suppose that $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = \lim_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) < +\infty$.

Lemma 1 ensures that $v'_\varepsilon \rightharpoonup v'$, $\nabla_y v_\varepsilon \rightharpoonup \nabla_y v$ in $L^2(Q)$, and $v \in H^1(Q)$. Since α' is a bounded function,

$$v'_\varepsilon + \langle \nabla_y v_\varepsilon, Ry \rangle \alpha' \rightharpoonup v' + \langle \nabla_y v, Ry \rangle \alpha'$$

in $L^2(Q)$. Then,

$$\begin{aligned} \int_Q |v' + \langle \nabla_y v, Ry \rangle \alpha'(s)|^2 ds dy &\leq \liminf_{\varepsilon \rightarrow 0} \int_Q |v'_\varepsilon + \langle \nabla_y v_\varepsilon, Ry \rangle \alpha'(s)|^2 ds dy \\ &\leq \liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon). \end{aligned}$$

By Lemma 2, we can write $v(s, y) = w(s)u_n(y)$ with $w \in H^1(I)$. Thus,

$$b_n(w) = b_n(v) \leq \liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon),$$

and item (i) is proven.

To prove (ii), we are going to show that for each $v \in \mathcal{H}_n$ there exists a sequence $(v_\varepsilon)_\varepsilon$ in \mathcal{H}_n so that $v_\varepsilon \rightarrow v$ in \mathcal{H}_n and $\lim_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = b_n(v)$. At first, consider the particular case $v(s, y) = w(s)u_n(y)$ with $w \in H^1(I)$. Take $v_\varepsilon := v$, for all $\varepsilon > 0$. Note that $b_n^\varepsilon(v) = b_n(w)$, for all $\varepsilon > 0$, and

$$\lim_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = b_n(v).$$

On the other hand, if $v \in \mathcal{H}_n \setminus \{w(s)u_n(y) : w \in H^1(I)\}$, one has $b_n(v) = +\infty$. Let $(v_\varepsilon)_\varepsilon$ be an arbitrary sequence so that $v_\varepsilon \rightarrow v$ in \mathcal{H}_n . In this case, $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = +\infty$. In fact, if we suppose that $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) < +\infty$, by Lemmas 1 and 2 we should have $v = wu_n$, with $w \in H^1(I)$, but this is not true. Therefore, $+\infty = \liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = \lim_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = b_n(v)$. Then, item (ii) is satisfied. \square

Proposition 2. *For each $n \in \mathbb{N}$, the sequence of quadratic forms b_n^ε weakly Γ -converges to b_n , as $\varepsilon \rightarrow 0$.*

Proof. At first, we are going to show the condition (i) of the definition of weak Γ -convergence, i.e., $b_n(v) \leq \liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon)$, for all sequence $v_\varepsilon \rightharpoonup v$ in \mathcal{H}_n . So, assume the weak convergence $v_\varepsilon \rightharpoonup v$. Initially, consider the case where $(v_\varepsilon)_\varepsilon$ does not belong to $\mathcal{H}_n \cap H^1(Q)$. Then, $b_n^\varepsilon(v_\varepsilon) = +\infty$, for all $\varepsilon > 0$, and the inequality is proven. Now, assume that $(v_\varepsilon)_\varepsilon \subset \mathcal{H}_n \cap H^1(Q)$. Suppose that $v = wu_n$, with $w \in H^1(I)$. By definition, $b_n(v) < +\infty$. If $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = +\infty$ the inequality is proven. Now, suppose that $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) < +\infty$. Passing to a subsequence if necessary, we can suppose that $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = \lim_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) < +\infty$. As in the proof of Proposition 1,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) &\geq \int_Q |v' + \langle \nabla_y v, Ry \rangle \alpha'(s)|^2 ds dy \\ &= b_n(w). \end{aligned}$$

Now, suppose that v does not belong to the subspace $\{wu_n : w \in H^1(I)\}$. We are going to show that necessarily $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = +\infty$. In fact, let P_{n+1} be the orthogonal projection onto \mathcal{H}_{n+1} . We have $\|P_{n+1}v\| > 0$. Since $v_\varepsilon \rightharpoonup v$ in $\mathcal{H}_n \cap H^1(Q)$, $P_{n+1}v_\varepsilon \rightharpoonup P_{n+1}v$ and

$$\liminf_{\varepsilon \rightarrow 0} \|P_{n+1}v_\varepsilon\| \geq \|P_{n+1}v\| > 0. \quad (16)$$

Note that

$$b_n^\varepsilon(v_\varepsilon) \geq \frac{1}{\varepsilon^2} \int_Q (|\nabla_y v_\varepsilon|^2 - \lambda_n |v_\varepsilon|^2) \, ds dy. \quad (17)$$

The strategy is to estimate the term on the right side of this inequality.

For $\psi \in H^1(S) \cap \mathcal{W}_{n-1}$, denote by ψ^n the component of ψ in $[u_n]$ and by Q_{n+1} the orthogonal projection onto \mathcal{W}_n in $H^1(S)$. Thus,

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_Q (|\nabla_y v_\varepsilon|^2 - \lambda_n |v_\varepsilon|^2) \, ds dy &= \frac{1}{\varepsilon^2} \int_I \left(\|\nabla_y v_\varepsilon(s, \cdot)\|_{L^2(S)}^2 - \lambda_n \|v_\varepsilon(s, \cdot)\|_{L^2(S)}^2 \right) ds \\ &= \frac{1}{\varepsilon^2} \int_I \left(\|v_\varepsilon(s, \cdot)\|_{H^1(S)}^2 - (\lambda_n + 1) \|v_\varepsilon(s, \cdot)\|_{L^2(S)}^2 \right) ds \\ &= \frac{1}{\varepsilon^2} \int_I \left(\|Q_{n+1}v_\varepsilon(s, \cdot)\|_{H^1(S)}^2 + \|v_\varepsilon^n(s, \cdot)\|_{H^1(S)}^2 \right. \\ &\quad \left. - (\lambda_n + 1) \|v_\varepsilon(s, \cdot)\|_{L^2(S)}^2 \right) ds \\ &= \frac{1}{\varepsilon^2} \int_I \left(\|\nabla_y Q_{n+1}v_\varepsilon(s, \cdot)\|_{L^2(S)}^2 + \|Q_{n+1}v_\varepsilon(s, \cdot)\|_{L^2(S)}^2 \right. \\ &\quad \left. + \|\nabla_y v_\varepsilon^n(s, \cdot)\|_{L^2(S)}^2 + \|v_\varepsilon^n(s, \cdot)\|_{L^2(S)}^2 \right. \\ &\quad \left. - (\lambda_n + 1) \|v_\varepsilon(s, \cdot)\|_{L^2(S)}^2 \right) ds \\ &\geq \frac{1}{\varepsilon^2} \int_I \left(\lambda_{n+1} \|Q_{n+1}v_\varepsilon(s, \cdot)\|_{L^2(S)}^2 + \lambda_n \|v_\varepsilon^n(s, \cdot)\|_{H^1(S)}^2 \right. \\ &\quad \left. - \lambda_n \|v_\varepsilon(s, \cdot)\|_{L^2(S)}^2 \right) ds \\ &= \frac{1}{\varepsilon^2} \int_I (\lambda_{n+1} - \lambda_n) |Q_{n+1}v_\varepsilon|^2 \, ds dy \\ &= \frac{(\lambda_{n+1} - \lambda_n)}{\varepsilon^2} \|P_{n+1}v_\varepsilon\|^2 \\ &\geq \frac{(\lambda_{n+1} - \lambda_n)}{\varepsilon^2} \|P_{n+1}v\|^2. \end{aligned}$$

This estimate, (16), (17) and the fact that $\lambda_{n+1} > \lambda_n$, imply that $\liminf_{\varepsilon \rightarrow 0} b_n^\varepsilon(v_\varepsilon) = +\infty$.

Finally, the condition (ii) of the definition of weak Γ -convergence can be proven in a similar way to the proof of Proposition 1. \square

Proof of Theorema 1: (A) This item follows by Propositions 1 and 2 of this section and Proposition 3 in Appendix A.2.

(B) We have to verify the itens $a)$, $b)$, and $c)$ of Propostion 4 in Appendix A.2. Item $a)$ follows by Propositions 1 and 2. It is known that the operator T_n has compact resolvent. Thus, $b)$ is satisfied. It remains to ensure $c)$. Consider the subspace $\mathcal{K} := \{\mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_{n-1}\}^\perp$. By Rellich-Kondrachov Theorem, \mathcal{K} is compactly embedded in \mathcal{H}_n . Thus, if $(v_\varepsilon)_\varepsilon$ is a bounded sequence in \mathcal{H}_n and $(b_n^\varepsilon(v_\varepsilon))_\varepsilon$ is also bounded, a similar proof to the Lemma

1 shows that $(v_\varepsilon)_\varepsilon$ is a bounded sequence in \mathcal{K} . So, item c) is satisfied. By Proposition 4 in Appendix A.2, T_n^ε converges in the norm resolvent sense to T_n in \mathcal{L}_n . By Corollary 2.3 in [6], we have the asymptotic behavior of the eigenvalues given by (13).

4 Spectral properties in the case of periodic waveguide

Consider $\Omega_\varepsilon^\alpha$ as in the Introduction in the particular case where $I = \mathbb{R}$ and $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 and periodic function, i.e., there exists $L > 0$ so that $\alpha(s + L) = \alpha(s)$, for all $s \in \mathbb{R}$. In this context, the goal of this section is to find spectral information about the spectrum of $-\Delta_n^\varepsilon$, for each $n \in \mathbb{N}$. Namely, we study the continuous absolutely spectrum $\sigma_{ac}(-\Delta_n^\varepsilon)$ and the existence and location of band gaps in $\sigma(-\Delta_n^\varepsilon)$.

4.1 Preliminary results

Due to the periodic characteristics of $-\Delta_n^\varepsilon$, to prove Theorems 2 and 3, we are going to use the Floquet-Bloch reduction under the Brillouin zone $\mathcal{C} = [-\pi/L, \pi/L)$. More precisely, define $Q_L := (0, L) \times S$, $\mathcal{L}_n^L := \{w(s)u_n(y) : w \in L^2(0, L)\}$, $n \in \mathbb{N}$,

$$\mathcal{H}_n^L := \begin{cases} L^2(Q_L), & n = 1, \\ (\mathcal{L}_1^L \oplus \mathcal{L}_2^L \oplus \cdots \oplus \mathcal{L}_{n-1}^L)^\perp, & n = 2, 3, \dots \end{cases}$$

Consider the family of quadratic forms acting in \mathcal{H}_n^L :

$$\hat{b}_n^\varepsilon(\theta)(\varphi) = \int_{Q_L} \left(|\varphi' + i\theta\varphi + \langle \nabla_y \varphi, Ry \rangle \alpha'(s)|^2 + \frac{|\nabla_y \varphi|^2}{\varepsilon^2} \right) ds dy, \quad \theta \in \mathcal{C}, \quad (18)$$

$\text{dom } \hat{b}_n^\varepsilon(\theta) = \{\varphi \in H^1(Q_L) \cap \mathcal{H}_n^L; \varphi(0, \cdot) = \varphi(L, \cdot) \text{ in } L^2(S)\}$. Denote by $-\Delta_n^\varepsilon(\theta)$ the self-adjoint operator associated with $\hat{b}_n^\varepsilon(\theta)$.

Lemma 3. *For each $n \in \mathbb{N}$, $\{-\Delta_n^\varepsilon(\theta), \theta \in \mathcal{C}\}$ is an analytic family of type (B).*

Proof. At first, note that $\text{dom } \hat{b}_n^\varepsilon(\theta)$ does not depend on θ . For each $\theta \in \mathcal{C}$, write $\hat{b}_n^\varepsilon(\theta) = \hat{b}_n^\varepsilon(0) + c_n^\varepsilon(\theta)$, where, for $\varphi \in \text{dom } \hat{b}_n^\varepsilon(0)$,

$$\begin{aligned} c_n^\varepsilon(\theta)(\varphi) &:= \hat{b}_n^\varepsilon(\theta)(\varphi) - \hat{b}_n^\varepsilon(0)(\varphi) \\ &= 2 \operatorname{Re} \left(\int_{Q_L} \overline{(\varphi' + \langle \nabla_y \varphi, Ry \rangle \alpha'(s))} (i\theta\varphi) ds dy \right) + \theta^2 \int_{Q_L} |\varphi|^2 ds dy. \end{aligned}$$

We affirm that $c_n^\varepsilon(\theta)$ is $\hat{b}_n^\varepsilon(0)$ -bounded with zero relative bound. In fact, given $\delta > 0$,

$$\begin{aligned} |c_n^\varepsilon(\theta)(\varphi)| &\leq 2 \int_{Q_L} |\varphi' + \langle \nabla_y \varphi, Ry \rangle \alpha'(s)| |i\theta\varphi| ds dy + \theta^2 \int_{Q_L} |\varphi|^2 ds dy \\ &\leq \delta \int_{Q_L} |\varphi' + \langle \nabla_y \varphi, Ry \rangle \alpha'(s)|^2 ds dy + \theta^2 (1/\delta + 1) \int_{Q_L} |\varphi|^2 ds dy \\ &\leq \delta \hat{b}_n^\varepsilon(0)(\varphi) + (\pi/L)^2 (1/\delta + 1) \|\varphi\|_{\mathcal{H}_n^L}^2, \end{aligned}$$

for all $\varphi \in \text{dom } \hat{b}_n^\varepsilon(0)$, for all $\theta \in \mathcal{C}$. Since $\delta > 0$ is arbitrary, the affirmation is proven. By Theorem 4.8, Chapter VII in [9], $\{\hat{b}_n^\varepsilon(\theta) : \theta \in \mathcal{C}\}$ is an analytic family of type (a). Consequently, $\{-\Delta_n^\varepsilon(\theta), \theta \in \mathcal{C}\}$ is an analytic family of type (B). \square

Lemma 4. *There exists a unitary operator $\mathcal{U}_n : \mathcal{H}_n \rightarrow \int_{\mathcal{C}}^{\oplus} \mathcal{H}_n^L d\theta$, so that,*

$$\mathcal{U}_n(-\Delta_n^\varepsilon)\mathcal{U}_n^{-1} = \int_{\mathcal{C}}^{\oplus} -\Delta_n^\varepsilon(\theta) d\theta.$$

Proof. For $(\theta, s, y) \in \mathcal{C} \times Q_L$, define

$$(\mathcal{U}_n f)(\theta, s, y) := \sum_{k \in \mathbb{Z}} \sqrt{\frac{L}{2\pi}} e^{-ikL\theta - i\theta s} f(s + Lk, y), \quad \text{dom } \mathcal{U}_n = \mathcal{H}_n,$$

which is a unitary operator onto $\int_{\mathcal{C}}^{\oplus} \mathcal{H}_n^L d\theta$; the definition of \mathcal{U}_n is based on [2, 10] .

Recall the quadratic form \bar{b}_n^ε ; see (6) in the Introduction. Consider

$$q_n^\varepsilon(\varphi) := \bar{b}_n^\varepsilon(\mathcal{U}_n^{-1}\varphi), \quad \text{dom } q_n^\varepsilon := \mathcal{U}_n(\text{dom } \bar{b}_n^\varepsilon).$$

Note that q_n^ε is a closed and bounded from below quadratic form in the Hilbert space $\int_{\mathcal{C}}^{\oplus} \mathcal{H}_n^L d\theta$, and $\mathcal{U}_n(-\Delta_n^\varepsilon)\mathcal{U}_n^{-1}$ is the self-adjoint operator associated with it.

For $(s, y) \in Q_L$ and $k \in \mathbb{Z}$,

$$(\mathcal{U}_n^{-1}\varphi)(s + Lk, y) = \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta + is\theta} \varphi(\theta, s, y) d\theta,$$

$$(\mathcal{U}_n^{-1}\varphi)'(s + Lk, y) = \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta + is\theta} (\varphi'(\theta, s, y) + i\theta\varphi(\theta, s, y)) d\theta,$$

and

$$\nabla_y(\mathcal{U}_n^{-1}\varphi)(s + Lk, y) = \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta + is\theta} \nabla_y \varphi(\theta, s, y) d\theta.$$

Since α' is an L -periodic function, by Parseval's identity, and by Fubini's Theorem, we

have

$$\begin{aligned}
q_n^\varepsilon(\varphi) &= \bar{b}_n^\varepsilon(\mathcal{U}_n^{-1}\varphi) \\
&= \int_Q \left(|(\mathcal{U}_n^{-1}\varphi)' + \langle \nabla_y(\mathcal{U}_n^{-1}\varphi), Ry \rangle \alpha'(s)|^2 + \frac{|\nabla_y(\mathcal{U}_n^{-1}\varphi)|^2}{\varepsilon^2} \right) ds dy \\
&= \sum_{k \in \mathbb{Z}} \int_{Q_L} |(\mathcal{U}_n^{-1}\varphi)'(s + Lk, y) + \langle \nabla_y(\mathcal{U}_n^{-1}\varphi)(s + Lk, y), Ry \rangle \alpha'(s)|^2 ds dy \\
&+ \sum_{k \in \mathbb{Z}} \int_{Q_L} \frac{1}{\varepsilon^2} |\nabla_y(\mathcal{U}_n^{-1}\varphi)(s + Lk, y)|^2 ds dy \\
&= \int_{Q_L} \sum_{k \in \mathbb{Z}} \left| \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta + is\theta} (\varphi'(\theta, s, y) + i\theta\varphi(\theta, s, y) + \langle \nabla_y\varphi(\theta, s, y), Ry \rangle \alpha'(s)) d\theta \right|^2 ds dy \\
&+ \int_{Q_L} \sum_{k \in \mathbb{Z}} \frac{1}{\varepsilon^2} \left| \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta + is\theta} \nabla_y\varphi(\theta, s, y) d\theta \right|^2 ds dy \\
&= \int_{Q_L} \left(\int_{\mathcal{C}} |(\varphi'(\theta, s, y) + i\theta\varphi(\theta, s, y) + \langle \nabla_y\varphi(\theta, s, y), Ry \rangle \alpha'(s))|^2 d\theta \right) ds dy \\
&+ \int_{Q_L} \left(\int_{\mathcal{C}} \frac{1}{\varepsilon^2} |\nabla_y\varphi(\theta, s, y)|^2 d\theta \right) ds dy \\
&= \int_{\mathcal{C}} \left(\int_{Q_L} |(\varphi'(\theta, s, y) + i\theta\varphi(\theta, s, y) + \langle \nabla_y\varphi(\theta, s, y), Ry \rangle \alpha'(s))|^2 ds dy \right) d\theta \\
&+ \int_{\mathcal{C}} \left(\int_{Q_L} \frac{1}{\varepsilon^2} |\nabla_y\varphi(\theta, s, y)|^2 ds dy \right) d\theta \\
&=: \int_{\mathcal{C}} \hat{b}_n^\varepsilon(\theta)(\varphi(\theta)) d\theta.
\end{aligned}$$

Then, $\varphi \in \text{dom } q_n^\varepsilon$ if, and only if, $\varphi \in \int_{\mathcal{C}}^\oplus \mathcal{H}_n^L d\theta$ and $\varphi(\theta) \in \text{dom } \hat{b}_n^\varepsilon(\theta)$, a.e. θ .

Now, consider the self-adjoint operator

$$Q_n^\varepsilon := \int_{\mathcal{C}}^\oplus -\Delta_n^\varepsilon(\theta) d\theta,$$

where

$$\text{dom } Q_n^\varepsilon := \left\{ \varphi : \varphi(\theta) \in \text{dom } (-\Delta_n^\varepsilon(\theta)), \text{ a.e. } \theta; \int_{\mathcal{C}} \| -\Delta_n^\varepsilon(\theta)\varphi(\theta) \|_{\mathcal{H}_n^L}^2 d\theta < +\infty \right\}.$$

For each $\varphi \in \text{dom } q_n^\varepsilon$ and for each $\eta \in \text{dom } Q_n^\varepsilon$,

$$\begin{aligned}
q_n^\varepsilon(\varphi, \eta) &= \int_{\mathcal{C}} \hat{b}_n^\varepsilon(\theta) (\varphi(\theta), \eta(\theta)) d\theta \\
&= \int_{\mathcal{C}} \langle \varphi(\theta), -\Delta_n^\varepsilon(\theta)\eta(\theta) \rangle_{\mathcal{H}_n^L} d\theta \\
&= \int_{\mathcal{C}} \langle \varphi(\theta), (Q_n^\varepsilon \eta)(\theta) \rangle_{\mathcal{H}_n^L} d\theta \\
&= \langle \varphi, Q_n^\varepsilon \eta \rangle.
\end{aligned}$$

Therefore, Q_n^ε is the self-adjoint operator associated with q_n^ε and, by uniqueness, $Q_n^\varepsilon = \mathcal{U}_n(-\Delta_n^\varepsilon)\mathcal{U}_n^{-1}$. \square

4.2 Proof of Theorem 2

Since each $-\Delta_n^\varepsilon(\theta)$ has compact resolvent and is lower bounded, its spectrum is discrete. We denote by $E_{n,j}(\varepsilon, \theta)$ the j th eigenvalue of $-\Delta_n^\varepsilon(\theta)$, counted with multiplicity, and by $\psi_{n,j}(\varepsilon, \theta)$ the corresponding normalized eigenfunction, i.e.,

$$-\Delta_n^\varepsilon(\theta)\psi_{n,j}(\varepsilon, \theta) = E_{n,j}(\varepsilon, \theta)\psi_{n,j}(\varepsilon, \theta), \quad j = 1, 2, 3, \dots, \quad \theta \in \mathcal{C}.$$

We have

$$E_{n,1}(\varepsilon, \theta) \leq E_{n,2}(\varepsilon, \theta) \leq \dots \leq E_{n,j}(\varepsilon, \theta) \leq \dots, \quad \theta \in \mathcal{C},$$

$$\sigma(-\Delta_n^\varepsilon) = \cup_{j=1}^{\infty} \{E_{n,j}(\varepsilon, \mathcal{C})\}, \quad \text{where } E_{n,j}(\varepsilon, \mathcal{C}) := \{E_{n,j}(\varepsilon, \theta) : \theta \in \mathcal{C}\};$$

each $E_{n,j}(\varepsilon, \mathcal{C})$ is called of the j th band of $\sigma(-\Delta_n^\varepsilon)$.

Lemma 3 ensures that the functions $E_{n,j}(\varepsilon, \theta)$ are real analytic functions in θ ; consequently, each $E_{n,j}(\varepsilon, \mathcal{C})$ is either a closed interval or a one point set. The goal is to find an asymptotic behavior for the eigenvalues $E_{n,j}(\varepsilon, \theta)$, as $\varepsilon \rightarrow 0$.

Based on the discussion in the Introduction, we start to study the sequence

$$b_n^\varepsilon(\theta)(\psi) := \hat{b}_n^\varepsilon(\theta)(\psi) - \frac{\lambda_n}{\varepsilon^2} \|\psi\|_{\mathcal{H}_n^L}^2, \quad (19)$$

$\text{dom } b_n^\varepsilon(\theta) := \text{dom } \hat{b}_n^\varepsilon(\theta)$. The self-adjoint operator associated with $b_n^\varepsilon(\theta)$ is $T_n^\varepsilon(\theta) := -\Delta_n^\varepsilon(\theta) - (\lambda_n/\varepsilon^2)\mathbf{1}$.

Define the one-dimensional quadratic form

$$\begin{aligned} b_n(\theta)(w) &:= b_n^\varepsilon(\theta)(wu_n) \\ &= \int_{Q_L} |w'u_n + i\theta wu_n + \langle \nabla_y u_n, Ry \rangle \alpha'(s)w|^2 ds dy, \end{aligned}$$

$\text{dom } b_n(\theta) := \{w \in H^1(0, L) : w(0) = w(L)\}$. Denote by $T_n(\theta)$ the self-adjoint operator associated with it. Namely,

$$T_n(\theta)w := (-i\partial_s + \theta)^2 w + V_n w,$$

$\text{dom } T_n(\theta) = \{w \in H^2(0, L) : w(0) = w(L), w'(0) = w'(L)\}$, where V_n is defined by (10) in the Introduction. We have

Theorem 4. *For each $n \in \mathbb{N}$ and each $\theta \in \mathcal{C}$ fixed, the sequence of self-adjoint operators $T_n^\varepsilon(\theta)$ converges in the norm resolvent sense to $T_n(\theta)$ in \mathcal{L}_n^L , as $\varepsilon \rightarrow 0$. Furthermore, for $n \in \mathbb{N}$, $j \in \mathbb{N}$ and $\theta \in \mathcal{C}$ fixed, one has*

$$\lim_{\varepsilon \rightarrow 0} \left(E_{n,j}(\varepsilon, \theta) - \frac{\lambda_n}{\varepsilon^2} \right) = k_{n,j}(\theta).$$

The proof of Theorem 4 is very similar to the proof of Theorem 1; it will be omitted here.

Denote by $k_{n,j}(\theta)$ the j th eigenvalue (counted multiplicity) of $T_n(\theta)$. As a consequence of Theorem 4, we have

Corollary 1. *For each $n \in \mathbb{N}$ and each $j \in \mathbb{N}$ fixed, one has*

$$\lim_{\varepsilon \rightarrow 0} \left(E_{n,j}(\varepsilon, \theta) - \frac{\lambda_n}{\varepsilon^2} \right) = k_{n,j}(\theta), \quad (20)$$

uniformly in \mathcal{C} .

Proof. For $n \in \mathbb{N}$ fixed, extend $b_n^\varepsilon(\theta)$ by the formulas (18) and (19), for all $\theta \in \overline{\mathcal{C}}$. Theorem 4 holds true if we consider $\overline{\mathcal{C}}$ instead of \mathcal{C} . Then, (20) holds true for each $j \in \mathbb{N}$ and each $\theta \in \overline{\mathcal{C}}$. On the other hand, if $\varepsilon_1 < \varepsilon_2$, then $b_n^{\varepsilon_2}(\theta)(\psi) \leq b_n^{\varepsilon_1}(\theta)(\psi)$, for all $\psi \in \text{dom } b_n^\varepsilon(\theta)$, for all $\theta \in \overline{\mathcal{C}}$. Thus, for each $j \in \mathbb{N}$ and each $\theta \in \overline{\mathcal{C}}$, the sequence $(E_{n,j}(\varepsilon, \theta) - \lambda_n/\varepsilon^2)$ is decreasing in ε . Now, the result follows by Dini's Theorem. \square

Proof of Theorem 2: Let $E > 0$, without loss of generality, we can suppose that, for all $\theta \in \mathcal{C}$, the spectrum of $-\Delta_n^\varepsilon(\theta)$ below E consists of exactly j_0 eigenvalues $\{E_{n,j}(\varepsilon, \theta)\}_{j=1}^{j_0}$. Lemma 3 ensures that $E_{n,j}(\varepsilon, \theta)$ and $\psi_{n,j}(\varepsilon, \theta)$ are real analytic functions in $\theta \in \mathcal{C}$.

Theorem XIII in [10] implies that the functions $k_{n,j}(\theta)$ are nonconstant. By Corollary 1, there exist $\varepsilon_E > 0$, $K(\varepsilon) > 0$, so that, $|E_{n,j}(\varepsilon, \theta) - (\lambda_n/\varepsilon^2) - \kappa_{n,j}(\theta)| < K(\varepsilon)$, for all $\theta \in \mathcal{C}$, for all $\varepsilon \in (0, \varepsilon_E)$, for all $j = 1, 2, \dots, j_0$, and $K(\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$. Consequently, the functions $E_{n,j}(\varepsilon, \theta)$ are nonconstant. Note that $\varepsilon_E > 0$ depends on j_0 , i.e., the thickness of the tube depends on the length of the energies to be covered. Now, by Section XIII.16 in [10], the conclusion follows.

4.3 Existence of band gaps

In this section we are going to prove Theorem 3. Consider the one-dimensional operator

$$\tilde{T}_n w := -w'' + V_n w, \quad \text{dom } \tilde{T}_n = H^2(\mathbb{R}).$$

We have denoted by $k_{n,j}(\theta)$ the j th eigenvalue (counted with multiplicity) of the operator $T_n(\theta)$. For each $j \in \mathbb{N}$, $k_{n,j}(\theta)$ is a real analytic function in \mathcal{C} . By Chapter XIII.16 in [10], we have the following properties:

- (a) $k_{n,j}(\theta) = k_{n,j}(-\theta)$ for all $\theta \in \mathcal{C}$, $j = 1, 2, 3, \dots$.
- (b) For j odd (resp. even), $k_{n,j}(\theta)$ is strictly monotone increasing (resp. decreasing) as θ increases from 0 to π/L . In particular,

$$\begin{aligned} k_{n,1}(0) < k_{n,1}(\pi/L) &\leq k_{n,2}(\pi/L) < k_{n,2}(0) \leq \dots \leq k_{n,2j-1}(0) < k_{n,2j-1}(\pi/L) \\ &\leq k_{n,2j}(\pi/L) < k_{n,2j}(0) \leq \dots \end{aligned}$$

For each $j \in \mathbb{N}$, define

$$B_{n,j} := \begin{cases} [k_{n,j}(0), k_{n,j}(\pi/L)], & \text{for } j \text{ odd,} \\ [k_{n,j}(\pi/L), k_{n,j}(0)], & \text{for } j \text{ even,} \end{cases}$$

and

$$G_{n,j} := \begin{cases} (k_{n,j}(\pi/L), k_{n,j+1}(\pi/L)), & \text{for } j \text{ odd so that } k_{n,j}(\pi/L) \neq k_{n,j+1}(\pi/L), \\ (k_{n,j}(0), k_{n,j+1}(0)), & \text{for } j \text{ even so that } k_{n,j}(0) \neq k_{n,j+1}(0), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, by Theorem XIII.90 in [10], one has $\sigma(\tilde{T}_n) = \cup_{j=1}^{\infty} B_{n,j}$, where $B_{n,j}$ is called of the j th band of $\sigma(\tilde{T}_n)$. If $G_{n,j} \neq \emptyset$, $G_{n,j}$ is called of gap of $\sigma(\tilde{T}_n)$.

By Corollary 1 and since $E_{n,j}(\varepsilon, \theta)$ is a decreasing sequence, for each $j \in \mathbb{N}$, and for each $\varepsilon > 0$,

$$\max_{\theta \in \mathcal{C}} E_{n,j}(\varepsilon, \theta) \leq \begin{cases} \lambda_n/\varepsilon^2 + k_{n,j}(\pi/L), & \text{for } j \text{ odd,} \\ \lambda_n/\varepsilon^2 + k_{n,j}(0), & \text{for } j \text{ even,} \end{cases}.$$

If $G_{n,j} \neq \emptyset$, again by Corollary 1, there exists $\varepsilon_j > 0$, so that, for all $\varepsilon \in (0, \varepsilon_j)$,

$$\min_{\theta \in \mathcal{C}} E_{n,j+1}(\varepsilon, \theta) \geq \begin{cases} \lambda_n/\varepsilon^2 + k_{n,j+1}(\pi/L) - |G_{n,j}|/2, & \text{for } j \text{ odd,} \\ \lambda_n/\varepsilon^2 + k_{n,j+1}(0) - |G_{n,j}|/2, & \text{for } j \text{ even,} \end{cases}$$

where $|\cdot|$ denotes the Lebesgue measure. Thus, we have

Corollary 2. *If $G_{n,j} \neq \emptyset$, there exists $\varepsilon_j > 0$, so that, for all $\varepsilon \in (0, \varepsilon_j)$,*

$$\min_{\theta \in \mathcal{C}} E_{n,j+1}(\varepsilon, \theta) - \max_{\theta \in \mathcal{C}} E_{n,j}(\varepsilon, \theta) \geq \frac{1}{2}|G_{n,j}|.$$

Another important tool to prove Theorem 3 is the following result due to Borg [3].

Theorem 5. (Borg) *Suppose that W is a real-valued, piecewise continuous function on $[0, L]$. Let μ_j^\pm be the j th eigenvalue of the following operator counted multiplicity respectively*

$$T^\pm := -\frac{d^2}{ds^2} + W(s), \quad \text{in } L^2(0, L),$$

with domain

$$\{w \in H^2(0, L) : w(0) = \pm w(L), w'(0) = \pm w'(L)\}.$$

We suppose that

$$\mu_j^+ = \mu_{j+1}^+, \quad \text{for all even } j,$$

and

$$\mu_j^- = \mu_{j+1}^-, \quad \text{for all odd } j.$$

Then, W is constant on $[0, L]$.

Proof of Theorem 3: Take $W(s) = V_n(s)$ in Theorem 5. The operator $T_n(0)$ (resp. $T_n(\pi/L)$) is unitarily equivalent to T^+ (resp. T^-); in fact, just to consider the unitary operator $(u_\theta w)(s) := e^{-i\theta s}w(s)$ with $\theta = 0$ (resp. $\theta = \pi/L$). Remember that $\{k_{n,j}(0)\}_{j \in \mathbb{N}}$ (resp. $\{k_{n,j}(\pi/L)\}_{j \in \mathbb{N}}$) are the eigenvalues of $T_n(0)$ (resp. $T_n(\pi/L)$).

Since $V_n(s)$ is not a constant function in $[0, L]$, by Borg's Theorem, without loss of generality, we can affirm that there exists $j \in \mathbb{N}$ so that $k_{n,j}(0) \neq k_{n,j+1}(0)$. Now, the result follows by Corollary 2.

4.4 Location of band gaps

In this section we find a location in $\sigma(-\Delta_n^\varepsilon)$ where Theorem 3 holds true. For this purpose, we use the scaling

$$\alpha \mapsto \gamma\alpha, \tag{21}$$

where $\gamma > 0$ is a small parameter. Thus, we obtain the waveguide $\Omega_{\varepsilon, \gamma}^\alpha := \Omega_\varepsilon^{\gamma\alpha}$. Consider $-\Delta_{\Omega_{\varepsilon, \gamma}^\alpha}^N$ instead of $-\Delta_{\Omega_\varepsilon^\alpha}^N$ in the Introduction. Denote by $\bar{b}_n^{\varepsilon, \gamma}$ and $\hat{b}_n^{\varepsilon, \gamma}(\theta)$ the quadratic forms obtained by replacing (21) in (6) and (18), respectively. The self-adjoint operators associated with these quadratic forms are denoted by $-\Delta_n^{\varepsilon, \gamma}$ and $-\Delta_n^{\varepsilon, \gamma}(\theta)$, respectively. Denote by $E_{n,j}(\gamma, \varepsilon, \theta)$ the j th eigenvalue of $-\Delta_n^{\varepsilon, \gamma}(\theta)$ counted with multiplicity.

Define $W_n(s) := C_n^1(S)(\alpha'(s))^2$. Write $W_n(s)$ as a Fourier Series, i.e.,

$$W_n(s) = \sum_{j=-\infty}^{+\infty} \frac{1}{\sqrt{L}} w_n^j e^{2\pi j i s / L} \quad \text{in } L^2[0, L].$$

The sequence $\{w_n^j\}_{j=-\infty}^{\infty}$ is called of Fourier coefficients of W_n . Since W_n is a real function, $w_n^j = \overline{w_n^{-j}}$, for all $j \in \mathbb{Z}$. We have

Theorem 6. Suppose that $V_n(s)$ is not a constant function in $[0, L]$ and $W_n(s)$ is non null. Let $j \in \mathbb{N}$ so that $w_n^j \neq 0$. Then, there exist $\gamma > 0$ small enough, $\varepsilon_{n,j+1} > 0$ and $C_{n,j}(\gamma) > 0$, so that, for all $\varepsilon \in (0, \varepsilon_{n,j+1})$,

$$\min_{\theta \in \mathcal{C}} E_{n,j+1}(\gamma, \varepsilon, \theta) - \max_{\theta \in \mathcal{C}} E_{n,j}(\gamma, \varepsilon, \theta) \geq C_{n,j}(\gamma).$$

To prove Theorem 6 we are going to use a strategy adopted in [13]. Some steps will be omitted here and a more complete proof can be found in that work. In addition, our problem requires some more adjustments which will be explained in the next paragraphs.

Some technical details. Let $W \in L^2[0, L]$ be a real function. For $\beta \in \mathbb{C}$, consider the operators

$$T_\beta^+ w = -w'' + \beta W(s)w, \quad \text{and} \quad T_\beta^- w = -w'' + \beta W(s)w,$$

with domains given by

$$\text{dom } T_\beta^+ = \{w \in H^2(0, L) : w(0) = w(L), w'(0) = w'(L)\}, \quad (22)$$

$$\text{dom } T_\beta^- = \{w \in H^2(0, L) : w(0) = -w(L), w'(0) = -w'(L)\}, \quad (23)$$

respectively. Denote by $\{l_j^+(\beta)\}_{j \in \mathbb{N}}$ and $\{l_j^-(\beta)\}_{j \in \mathbb{N}}$ the eigenvalues of T_β^+ and T_β^- , respectively. For $\beta \in \mathbb{R}$ and $j \in \mathbb{N}$, define

$$\delta_j^+(\beta) := l_{2j+1}^+(\beta) - l_{2j}^+(\beta) \quad \text{and} \quad \delta_j^-(\beta) := l_{2j}^-(\beta) - l_{2j-1}^-(\beta).$$

Now,

$$\delta_{2j-1}(\beta) := \delta_j^-(\beta) \quad \text{and} \quad \delta_{2j}(\beta) := \delta_j^+(\beta).$$

Let $\{w^j\}_{j=-\infty}^{+\infty}$ be the Fourier coefficients of W ;

$$W(s) = \sum_{j=-\infty}^{+\infty} \frac{1}{\sqrt{L}} w^j e^{2\pi j i s / L} \quad \text{in } L^2[0, L],$$

where $w^j = \overline{w^{-j}}$, for all $j \in \mathbb{Z}$.

The next theorem gives an asymptotic behavior for $\delta_j(\beta)$, as $\beta \rightarrow 0$, in terms of the Fourier coefficients of W .

Theorem 7. For each $j \in \mathbb{N}$,

$$\delta_j(\beta) = \frac{2}{\sqrt{L}} |w^j| |\beta| + O(|\beta|^2), \quad \beta \rightarrow 0, \beta \in \mathbb{R}.$$

A detailed proof of Theorem 7 can be found in [13].

Auxiliary problem. For each $\gamma > 0$ and $\theta \in \mathcal{C}$, consider the one-dimensional quadratic form

$$s_n^\gamma(\theta)(w) := \int_0^L (|w' + i\theta w|^2 + \gamma^2 W_n(s)|w|^2) \, ds,$$

$\text{dom } s_n^\gamma(\theta) := \{w \in H^1(0, L) : w(0) = w(L)\}$. The self-adjoint operator associated with $s_n^\gamma(\theta)$ is given by

$$S_n^\gamma(\theta)w := (-i\partial_s + \theta)^2 w + \gamma^2 W_n(s)w,$$

$\text{dom } S_n^\gamma(\theta) := \{w \in H^2(0, L) : w(0) = w(L), w'(0) = w'(L)\}$. Denote by $\nu_{n,j}(\gamma, \theta)$ the j th eigenvalue of $S_n^\gamma(\theta)$ counted with multiplicity.

Now, consider

$$b_n^\gamma(\theta)(w) := b_n^{\varepsilon, \gamma}(\theta)(wu_n) = \int_0^L (|w' + i\theta w|^2 + V_n^\gamma(s)|w|^2) ds,$$

$\text{dom } b_n^\gamma(\theta) := \{w \in H^1(0, L) : w(0) = w(L)\}$, where $V_n^\gamma(s) := \gamma^2 W_n(s) - \gamma C_n^2(s) \alpha''(s)$. The self-adjoint operator associated with $b_n^\gamma(\theta)$ is

$$T_n^\gamma(\theta)w := (-i\partial_s + \theta)^2 w + V_n^\gamma(s)w,$$

$\text{dom } T_n^\gamma(\theta) := \{w \in H^2(0, L) : w(0) = w(L), w'(0) = w'(L)\}$. Denote by $k_{n,j}(\gamma, \theta)$ the j th eigenvalue of $T_n^\gamma(\theta)$ counted with multiplicity.

Take $c > \max\{\|V_n\|_\infty, \|W_n\|_\infty\}$. Some straightforward calculations show that there exists $K > 0$, so that,

$$|(b_n^\gamma(\theta) + c)(w) - (s_n^\gamma(\theta) + c)(w)| \leq K \gamma |(b_n^\gamma(\theta) + c)(w)|, \quad \forall w \in \text{dom } b_n^\gamma(\theta), \quad (24)$$

for all $\theta \in \mathcal{C}$, for all $\gamma > 0$ small enough.

Inequality (24), Theorem 2 in [1], and Corollary 2.3 in [6] imply

Corollary 3. *For each $j \in \mathbb{N}$, there exists $\gamma_j > 0$, so that, for all $\gamma \in (0, \gamma_j)$,*

$$k_{n,j}(\gamma, \theta) = \nu_{n,j}(\gamma, \theta) + O(\gamma),$$

uniformly in \mathcal{C} .

Some estimates. I. We define

$$G_{n,j}(\gamma) := \begin{cases} (k_{n,j}(\gamma, \pi/L), k_{n,j+1}(\gamma, \pi/L)), & \text{for } j \text{ odd so that } k_{n,j}(\gamma, \pi/L) \neq k_{n,j+1}(\gamma, \pi/L), \\ (k_{n,j}(\gamma, 0), k_{n,j+1}(\gamma, 0)), & \text{for } j \text{ even so that } k_{n,j}(\gamma, 0) \neq k_{n,j+1}(\gamma, 0), \\ \emptyset, & \text{otherwise.} \end{cases}.$$

Namely, if $G_{n,j}(\gamma) \neq \emptyset$, it is called of gap of the spectrum $\sigma(T_n^\gamma)$, where

$$T_n^\gamma w := -w'' + V_n^\gamma(s)w, \quad \text{dom } T_n^\gamma = H^2(\mathbb{R}).$$

Similarly to the considerations of Section 4.3 and Corollary 2, we have

Corollary 4. *If $G_{n,j}(\gamma) \neq \emptyset$, there exist $\gamma_j > 0$ and $\varepsilon_j > 0$, so that, for all $\gamma \in (0, \gamma_j)$ and for all $\varepsilon \in (0, \varepsilon_j)$,*

$$\min_{\theta \in \mathcal{C}} E_{n,j+1}(\gamma, \varepsilon, \theta) - \max_{\theta \in \mathcal{C}} E_{n,j}(\gamma, \varepsilon, \theta) \geq \frac{1}{2} |G_{n,j}(\gamma)|.$$

II. Now, we consider

$$\tilde{G}_{n,j}(\gamma) := \begin{cases} (\nu_{n,j}(\gamma, \pi/L), \nu_{n,j+1}(\gamma, \pi/L)), & \text{for } j \text{ odd so that } \nu_{n,j}(\gamma, \pi/L) \neq \nu_{n,j+1}(\gamma, \pi/L), \\ (\nu_{n,j}(\gamma, 0), \nu_{n,j+1}(\gamma, 0)), & \text{for } j \text{ even so that } \nu_{n,j}(\gamma, 0) \neq \nu_{n,j+1}(\gamma, 0), \\ \emptyset, & \text{otherwise.} \end{cases};$$

if $\tilde{G}_{n,j}(\gamma) \neq \emptyset$, it is called of gap of $\sigma(S_n^\gamma)$, where

$$S_n^\gamma w := -w'' + \gamma^2 W_n(s)w, \quad \text{dom } S_n^\gamma = H^2(\mathbb{R}).$$

As in the proof of Theorem 3, consider the unitary operator $(u_\theta w)(s) = e^{-i\theta s}w(s)$. We define the self-adjoint operators $\tilde{S}_n^\gamma(0) := u_0 S_n^\gamma(0) u_0^{-1}$ and $\tilde{S}_n^\gamma(\pi/L) := u_{\pi/L} S_n^\gamma(\pi/L) u_{\pi/L}^{-1}$, whose eigenvalues are given by $\{\nu_{n,j}(\gamma, 0)\}_{j \in \mathbb{N}}$ and $\{\nu_{n,j}(\gamma, \pi/L)\}_{j \in \mathbb{N}}$, respectively. Furthermore, the domains of these operators are given by (22) and (23), respectively. Thus, we can see that $|\tilde{G}_{n,j}(\gamma)| = \delta_j(\gamma^2)$, for all $j \in \mathbb{N}$, if we consider $\beta = \gamma^2$ and $W(s) = W_n(s)$ in Theorem 7.

With the notes of the previous paragraphs, we have conditions to prove the main theorem of this section.

Proof of Theorem 6: Recall that we have denoted by $\{w_n^j\}_{j=-\infty}^{+\infty}$ the Fourier coefficients of W_n . Since W_n isn't a constant function in $[0, L]$, there exists $j \in \mathbb{N}$, so that, $w_n^j \neq 0$. By Theorem 7,

$$|\tilde{G}_{n,j}(\gamma)| = \frac{2}{\sqrt{L}} \gamma^2 |w_n^j| + O(\gamma^4), \quad \gamma \rightarrow 0.$$

This estimate and Corollary 3 imply that $|G_{n,j}(\gamma)| > 0$, for all $\gamma > 0$ small enough. By Corollary 4, theorem is proven by taking $C_{n,j}(\gamma) := |G_{n,j}(\gamma)|/2 > 0$.

Remark 1. Since we suppose that $V_n(s)$ is a non null function in $[0, L]$, if $W_n(s) = 0$, for all $s \in \mathbb{R}$, one can consider $\tilde{W}_n(s) := C_n^2(S)\alpha''(s)$ instead of $W_n(s)$ in this subsection. All the previous results also hold true in this case; the proofs are similar and will not be presented here.

A Appendix

A.1 The self-adjoint operator associated with b_n

Recall the quadratic form

$$b_n(w) = \int_Q |w' u_n + \langle \nabla_y u_n, Ry \rangle \alpha'(s) w|^2 ds,$$

$\text{dom } b_n = H^1(I)$. The goal is to show that the operator T_n defined by (9), (10), (11) and (12) in the Introduction is the self-adjoint operator associated with b_n .

Consider the particular case where $I = (a, b)$ is a bounded interval. Some calculations show that

$$b_n(w) = \int_a^b (|w'|^2 + V_n(s)|w|^2) ds + C_n^2(S)\alpha'(b)|w(b)|^2 - C_n^2(S)\alpha'(a)|w(a)|^2.$$

Let $b_n(w, u)$ be the sesquilinear form associated with $b_n(w)$. We have

$$b_n(w, u) = \langle w, T_n u \rangle, \quad \forall w \in \text{dom } b_n, \forall u \in \text{dom } T_n.$$

Then, T_n is self-adjoint operator associated with b_n . The case $I = \mathbb{R}$ can be proven in a similar way.

A.2 Γ -convergence

Let H be a (real or complex) Hilbert space and $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. The sequence of quadratic functionals $f_\varepsilon : H \rightarrow \overline{\mathbb{R}}$ strongly Γ -converges to $f : H \rightarrow \overline{\mathbb{R}}$ (that is, $f_\varepsilon \xrightarrow{S\Gamma} f$) iff the following two conditions are satisfied:

i) For every $v \in H$ and every $v_\varepsilon \rightarrow v$ in H one has

$$\liminf_{\varepsilon \rightarrow 0} f_\varepsilon(v_\varepsilon) \geq f(v).$$

ii) For every $v \in H$, there exists a sequence $v_\varepsilon \rightarrow v$ in H such that

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(v_\varepsilon) = f(v).$$

If the strong convergence $v_\varepsilon \rightarrow v$ is replaced by the weak convergence $v_\varepsilon \rightharpoonup v$ in i) and ii), then one has a characterization of the weakly Γ -converge (i.e., $f_\varepsilon \xrightarrow{W\Gamma} f$).

The following result can be found in [4] where is proven the version for real Hilbert spaces; the generalization for complex Hilbert spaces is presented in [5].

Proposition 3. *Let $d_\varepsilon; d$ be positive (or uniformly lower bounded) closed sesquilinear forms in the Hilbert space H , and $D_\varepsilon; D$ the corresponding associated positive self-adjoint operators. Then, the following statements are equivalent:*

- a) $d_\varepsilon \xrightarrow{S\Gamma} d$ and, for each $\zeta \in H$, $d(\zeta) \leq \liminf_{\varepsilon \rightarrow 0} d_\varepsilon(\zeta_\varepsilon)$, $\forall \zeta_\varepsilon \rightarrow \zeta$ in H .
- b) $d_\varepsilon \xrightarrow{S\Gamma} d$ and $d_\varepsilon \xrightarrow{W\Gamma} d$.
- c) D_ε converges to D in the strong resolvent sense in $H_0 = \overline{\text{dom } D} \subset H$, that is,

$$\lim_{\varepsilon \rightarrow 0} R_{-\lambda}(D_\varepsilon)\zeta = R_{-\lambda}(D)P\zeta, \quad \forall \zeta \in H, \forall \lambda > 0,$$

where P is the orthogonal projection onto H_0 .

The following result is due to [5].

Proposition 4. *Let $d_\varepsilon, d \geq \beta > -\infty$ closed sesquilinear forms and $D_\varepsilon, D \geq \beta \mathbf{1}$ the corresponding associated self-adjoint operators, and let $\overline{\text{dom } D} = H_0 \subset H$. Assume that the following three conditions hold:*

- a) $d_\varepsilon \xrightarrow{S\Gamma} d$ and $d_\varepsilon \xrightarrow{W\Gamma} d$.
- b) The resolvent operator $R_{-\lambda}(D)$ is compact in H_0 for some real number $\lambda > |\beta|$.
- c) There exists a Hilbert space \mathcal{K} , compactly embedded in H , so that if the sequence (ψ_ε) is bounded in H and $(d_\varepsilon(\psi_\varepsilon))$ is also bounded, then (ψ_ε) is a bounded subset of \mathcal{K} .

Then, D_ε converges in norm resolvent sense to D in H_0 as $\varepsilon \rightarrow 0$.

Remark 2. In both propositions, the domain of D is not supposed to be dense in H but is required that $\text{rng } D \subset H_0$; we say that D is self-adjoint in H_0 .

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